The World's Worst Roads —
What Are Their Profiles?

Self-Supported Research by

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Conventional Road-Vehicle Dynamics Analysis

- Characterize road roughness by power spectral density, estimated from extensive measurements on freeways, main roads, and minor roads.
- Calculate power spectra of vehicle accelerations by applying transfer functions from dynamic equations of motion.
- Estimate ride comfort by comparing calculated curves of acceleration versus frequency with empirically derived curves from subjective passenger judgements in test rides.
- Improve vehicle design by changing vehicle transfer functions to reduce conflicts between calculated and empirically acceptable acceleration power spectral densities.

Shortcomings of Conventional Method

- Roads are effectively treated as a sequence of constant-wavelength washboards, with little consideration of superposition effects over the broad spectrum of typical roads.
- Consequently, no treatment is given for the familiar and severe discomfort from road sections that build up large vehicle accelerations or displacements via worst-case profiles.
- Little guidance is given for road construction; roads are simply taken as an immutable random input. Worst-case profiles are unknown.

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Proposed Convex Model Approach

- Characterize road profiles with sets of Fourier coefficients bounded by an ellipsoid in coefficient space, whose radii change with wavelength in a manner analogous to the measured power spectral density functions.
- Calculate for each set of vehicle parameters and speed the maximum acceleration (or displacement) for any Fourier coefficient set within the ellipsoid.
- Corresponding to the maxima are road profiles that cause these worst-case responses. Worst-case profiles should be avoided in road construction and included in test track construction.
- Improve vehicle design by selecting parameters that minimize worst-case responses.

Description of Road Roughness

Long history of measured road profiles described by power spectral density $S(\nu)$:

$$S(\nu) = \begin{cases} k\nu^{-2.5} & 0.01 \le \nu \le 10 \\ k(0.01)^{-2.5} & 0 < \nu < 0.01 \end{cases} \quad (1)$$

where ν is wave number in cycles/m, and coefficient k, in units $m^{0.5}$, was determined as follows for $S(\nu)$ in m^2/(cycles/m):

$$\begin{array}{lll} 3\times10^{-8} < k < 50\times10^{-8} & \bar{k} = 10\times10^{-8} & \text{freeways} \\ 3\times10^{-8} < k < 800\times10^{-8} & \bar{k} = 50\times10^{-8} & \text{main roads} \\ 50\times10^{-8} < k < 3000\times10^{-8} & \bar{k} = 500\times10^{-8} & \text{minor roads} \end{array}$$

- Conventional analyses ⇒ Treat road roughness as random.
- Present analysis ⇒ New use of spectral distribution.

On road interval $0, L$ long enough to include wavelengths of interest, the n^{th} component of road spectrum is

$$\delta_n(x) = c_n \cos\left(\frac{2\pi n x}{L} + \beta_n\right) \quad (2)$$

With vehicle traveling at velocity V, then $x = Vt$ and motion at contact with the road is

$$\delta_n(t) = c_n \cos\left(\frac{2\pi n V t}{L} + \beta_n\right) = c_n \cos(\omega_n t + \beta_n) \tag{3}$$

and the excitation frequency is $\omega_n = 2\pi n V/L$ radians/s or $f_n = nV/L = \nu V$ cps.

From the definition of power spectral density $S(f)$

$$c_n^2 = \frac{2S(\nu)}{L} = 2kL^{1.5}n^{-2.5} \tag{4}$$

Total excitation is the sum of Fourier components given by equation (3).

$$\delta(t) = \sum_{n=1}^{N} c_n \cos(n\theta + \beta_n) \tag{5}$$

in which

$$\theta = 2\pi V t/L \tag{6}$$

is introduced for convenience so θ ranges from 0 to 2π as the vehicle moves from $x = 0$ to $x = Vt = L$. Thus, in the following finite transform analysis over the interval 0,L it is helpful to envision that the vehicle moves around a circular test track of length L.

Vehicle Dynamics

Simple one-degree-of-freedom model: body mass m, suspension spring K that rides on the road surface, viscous damper C in parallel with K. Equation of motion:

$$m\ddot{y} + C\dot{y} + Ky = -m\ddot{\delta} \tag{7}$$

where $\delta(t)$ is motion from equation (5), and $y(t) = y_m(t) - \delta(t)$ is relative displacement between mass motion $y_m(t)$ and the road surface. With

$$\omega^2 = K/m, \qquad 2\zeta\omega = C/m \tag{8}$$

equation (7) becomes

$$\ddot{y} + 2\zeta\omega\dot{y} + \omega^2 y = -\ddot{\delta} \tag{9}$$

where ω is the undamped natural frequency and ζ is the fraction of critical damping.

Steady-state vehicle acceleration for the n^{th} component of road roughness is

$$\ddot{y}_m = c_n\omega_n^2 \left[\frac{1 + 4\zeta^2\Omega_n^2}{(1-\Omega_n^2)^2 + 4\zeta^2\Omega^2}\right]^{1/2} \cos(\omega_n t + \beta_n - \alpha_{\mathrm{a}n}) \tag{10}$$

with phase shift relative to the road input

$$\alpha_{\mathrm{a}n} = \tan^{-1}(2\zeta\Omega_n) \tag{11}$$

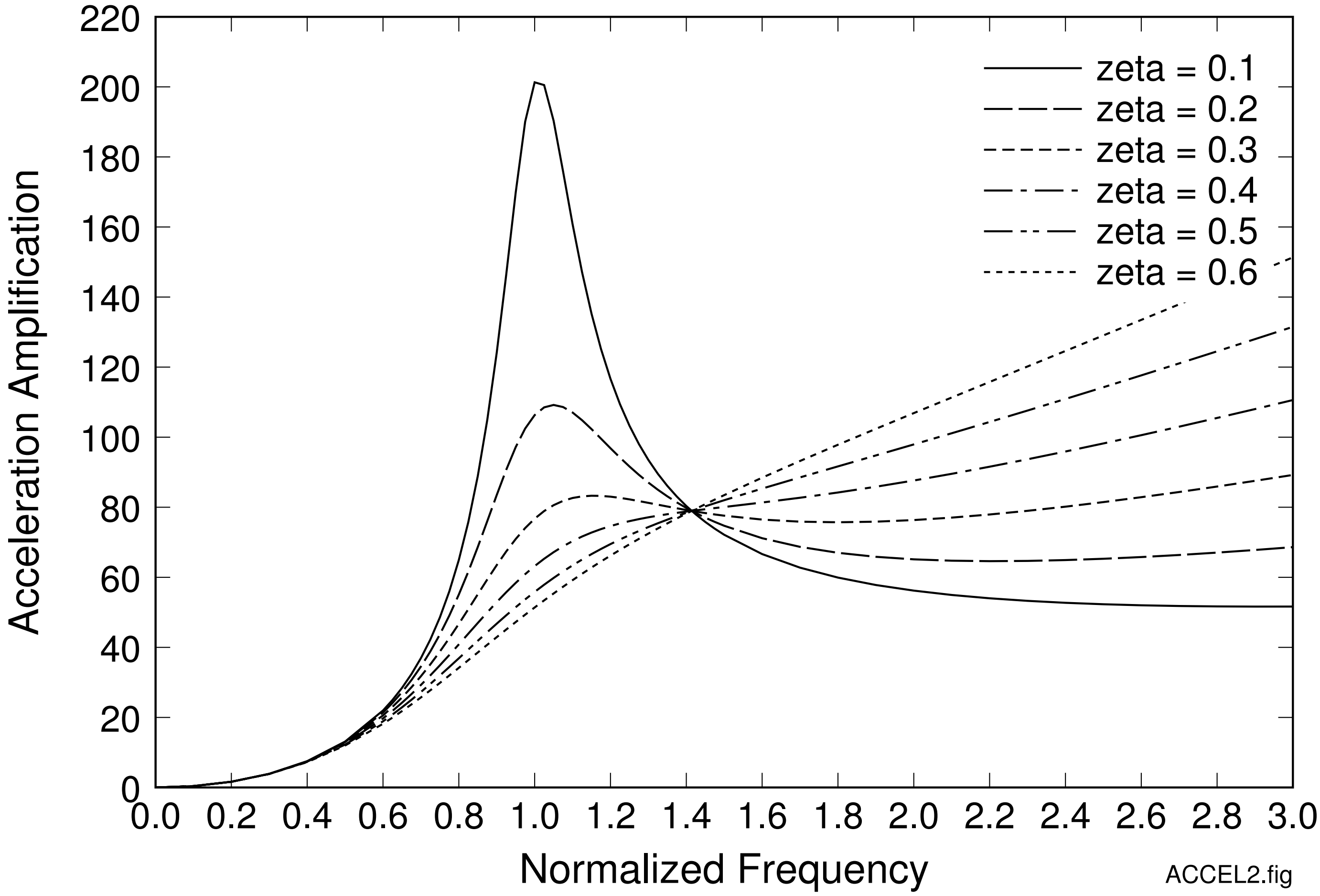


Figure 1. Acceleration amplification G_n (in s^{-2}) for $f = 1$ Hz.

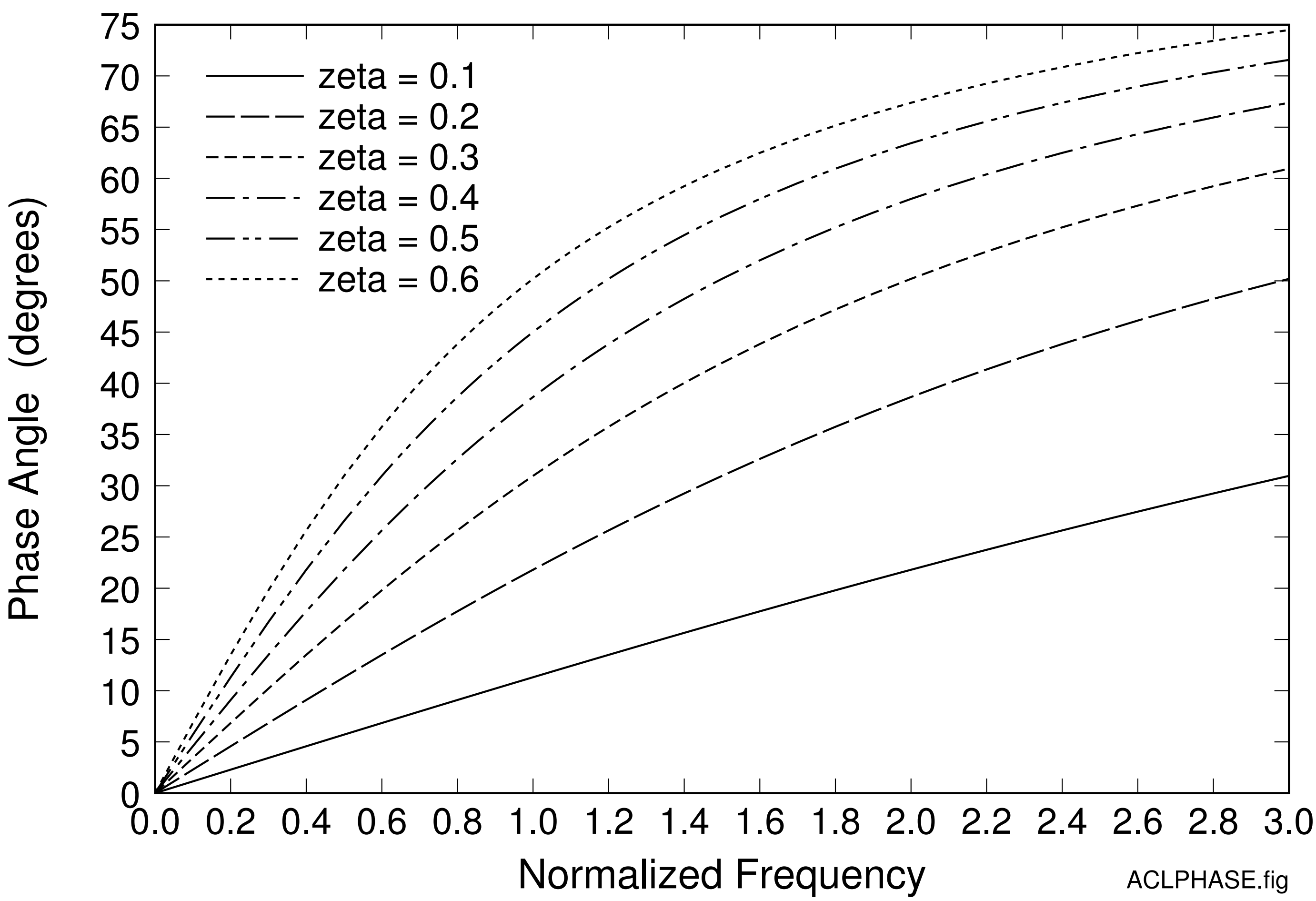


Figure 2. Phase angle α_{an} versus normalized frequency f_n/f.

The Convex Model

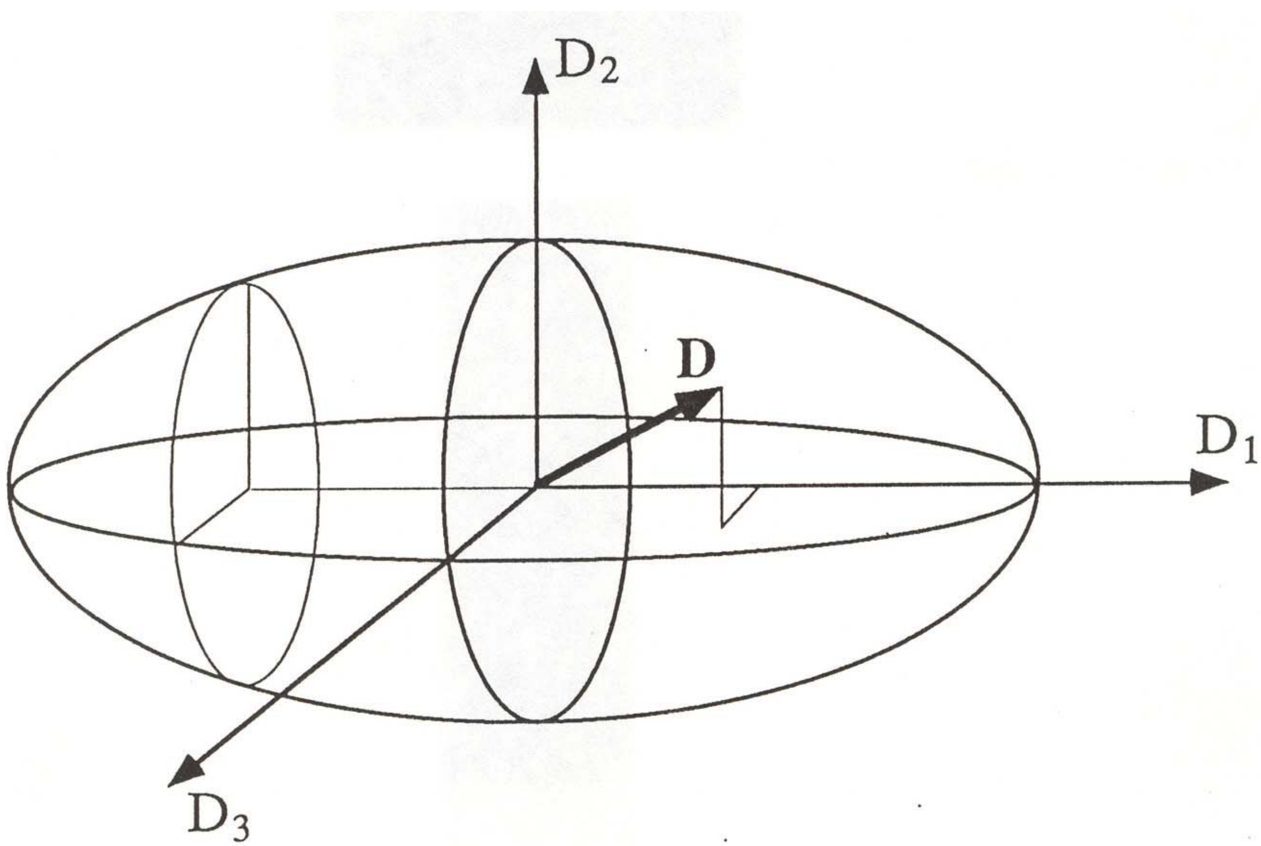
Convex bound: allowed range for each c_n^2 varies with n as in the power spectral density:

$$0 \leq c_n^2 \leq \kappa^2 n^{-2.5} \tag{12}$$

This is part of the more general specification: combinations of c_n will be taken such that they are bounded within an ellipsoid in c_n space:

$$\sum_{n=1}^{n=N} n^{2.5} c_n^2 \leq \kappa^2 \tag{13}$$

The ellipsoid extremes define the the ranges given by (12), found be setting $c_i = 0$ for $i \neq n$ in expression (13).



For convex analysis introduce vector representation for the road profile and vehicle responses. Then equation (5) becomes

$$\delta(t) = \sum_{n=1}^{n=N} c_n \cos(n\theta + \beta_n) \equiv \mathbf{D}^T \boldsymbol{\varphi} \tag{14}$$

where

$$\mathbf{D}^T = \{c_1,\ c_2,\ \cdots\ c_N\}$$

$$\boldsymbol{\varphi}^T = \{\cos(\theta + \beta_1),\ \cos(2\theta + \beta_2),\ \cdots\ \cos(N\theta + \beta_N)\}$$

Each wavelength of road roughness induces the steady-state responses given by (10). For maximum total response the modal responses must be in phase at time $t_{\max}$. For pleasing display of results, choose $t_{\max} = L/2V$, at which $\theta = \pi$. Thus, β_n is chosen such that, in (10), $\omega_n t + \beta_n - \alpha_n = 0$ at $\theta = \pi$. With $\omega_n t \equiv n\theta$ this gives

$$\beta_n = \alpha_n - n\pi, \quad \text{and hence} \quad \ddot{y}(\theta) = \sum_{n=1}^{n=N} c_n G_n \cos n(\theta - \pi) \tag{15}$$

In vector notation,

$$\ddot{y}(\theta) = \mathbf{D}^T \boldsymbol{\phi} \tag{16}$$

with

$$\boldsymbol{\phi}(\theta) = \mathbf{G}\boldsymbol{\varphi}_\pi(\theta), \quad \mathbf{G} = \mathrm{diag}\{G_1,\ G_2,\ \cdots\ G_n\}$$

Two closely related sets of road roughness Fourier components c_n constitute the convex model: set R (range) of allowed coefficients and set E of extreme values of R.

$$R(\kappa, \mathbf{W}) = \{\mathbf{D} : \ \mathbf{D}^T \mathbf{W} \mathbf{D} \leq \kappa^2\} \tag{17}$$

which reads: set R is the set of all values of Fourier coefficients $\mathbf{D}$ such that $\mathbf{D}^T \mathbf{W} \mathbf{D} \leq \kappa^2$. Here $\mathbf{W}$ is an $N \times N$ positive definite, real, symmetric matrix that specifies the shape of the ellipsoid in N-dimensional space, and κ^2 is a positive number that specifies its size. In the present model,

$$\mathbf{W} = \mathrm{diag}\{1^{2.5}, \ 2^{2.5}, \ \cdots \ N^{2.5}\}, \tag{18}$$

and the ellipsoid expression evaluates to expression (13). Because vehicle response is linear in c_n, and because $R(\kappa, \mathbf{W})$ is a convex set (all convex combinations of its extreme points E are members of R, and these combinations completely map R), the maximum response occurs on the set of extreme points $E(\kappa, \mathbf{W})$, given by

$$E(\kappa, \mathbf{W}) = \{\mathbf{D} : \ \mathbf{D}^T \mathbf{W} \mathbf{D} = \kappa^2\} \tag{19}$$

Maximum response is found by maximizing $\mathbf{D}^T\boldsymbol{\phi}$ subject to the constraint that $\mathbf{D}$ is in the extreme-point set E. The desired maximum is found with the Hamiltonian

$$H = \mathbf{D}^T\boldsymbol{\phi} + \lambda(\mathbf{D}^T\mathbf{W}\mathbf{D} - \kappa^2) \tag{20}$$

in which $\mathbf{D}^T\boldsymbol{\phi}$ is the function to be maximized and λ is a Lagrangian multiplier for the constraint. The extremum condition

$$0 = \frac{\partial H}{\partial \mathbf{D}} = \boldsymbol{\phi} + 2\lambda\,\mathbf{W}\mathbf{D} \tag{21}$$

gives maximum

$$\mathbf{D}^T\boldsymbol{\phi} = \kappa\left[\boldsymbol{\phi}^T\mathbf{W}^{-1}\boldsymbol{\phi}\right]^{1/2} \tag{22}$$

The roughness coefficients and worst-road profile that give this maximum response are

$$\mathbf{D}_{\mathrm{mr}} = \frac{\kappa\,\mathbf{W}^{-1}\boldsymbol{\phi}}{\left[\boldsymbol{\phi}^T\,\mathbf{W}^{-1}\boldsymbol{\phi}\right]^{1/2}} \qquad \delta_{\mathrm{w}}(\theta) = \mathbf{D}_{\mathrm{mr}}^T\boldsymbol{\varphi}_{\pi}(\theta) \tag{23}$$

With ϕ from (16) and $\theta = \pi$, the maximum acceleration response is

$$\ddot{y}_{\max} = \kappa \left[\sum_{n=1}^{n=N} n^{-2.5} G_n^2 \right]^{1/2} \tag{24}$$

The road profile that gives this acceleration is

$$\delta_{\mathrm{w}}(\theta) = \kappa \sum_{n=1}^{n=N} n^{-2.5} G_n \cos(n\theta + \beta_n) \;/\; \left[\sum_{n=1}^{n=N} n^{-2.5} G_n^2 \right]^{1/2} \tag{25}$$

This worst-road profile can be used, along with a very simple road profile measurement, to obtain values for the constant κ. The measurement, $\hat{\delta}$, is simply half the difference between the highest and lowest points over a section of road of length comparable to the span of the worst-road profile. Then κ is chosen such that the peak of the worst-road profile is equal to this measured profile bound. When κ is determined in this way, the peak of the maximum acceleration response is

$$\frac{\ddot{y}_{\max}}{\hat{\delta}} = \sum_{n=1}^{n=N} n^{-2.5} G_n^2 \;/\; \sum_{n=1}^{n=N} n^{-2.5} G_n \cos \alpha_{\mathrm{a}n} \tag{26}$$

in which $\theta = \pi$ was used in (25) to obtain the peak of the worst-road profile.

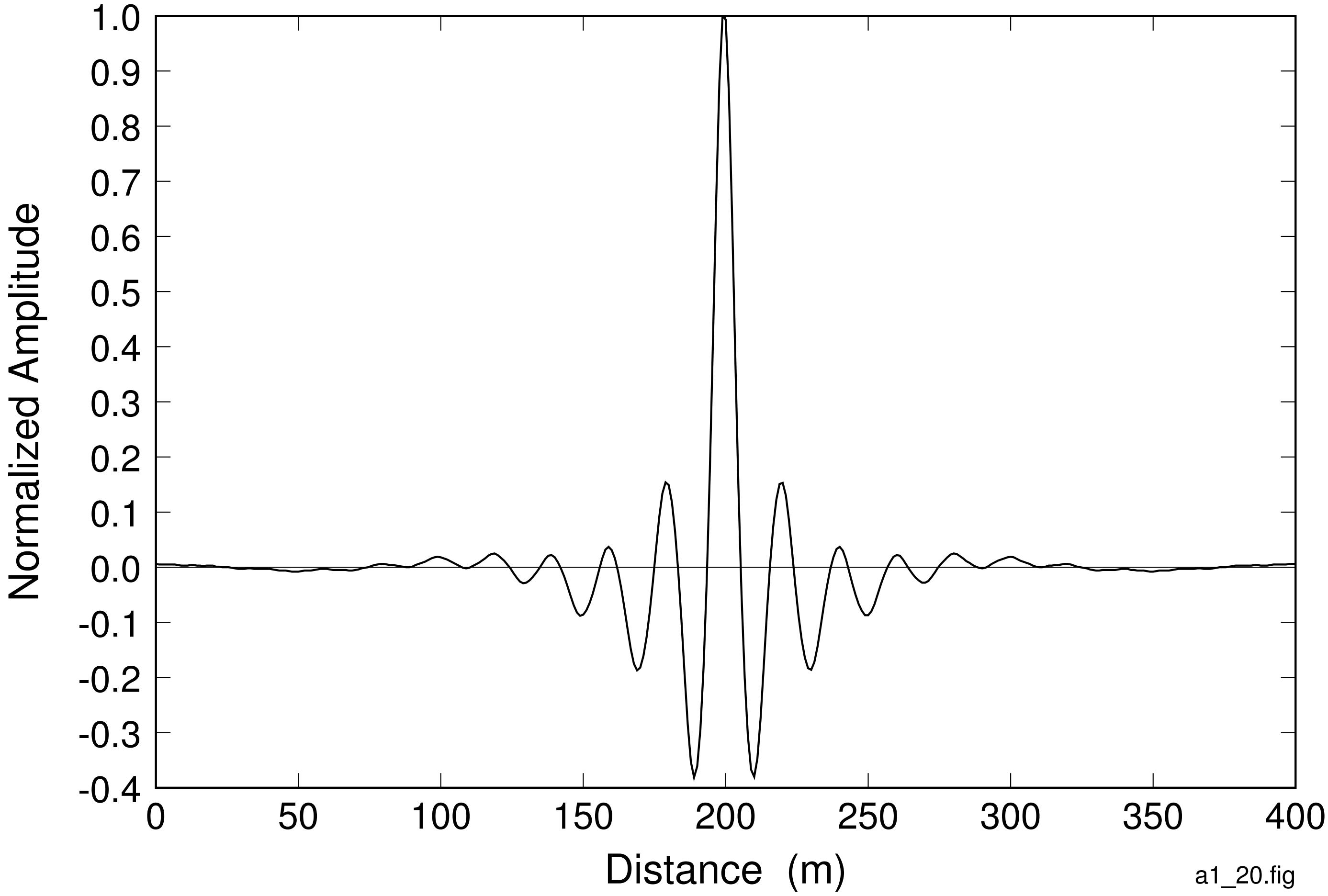


Figure 3. Worst-road for acceleration with $V = 20$ m/s and $\zeta = 0.1$.

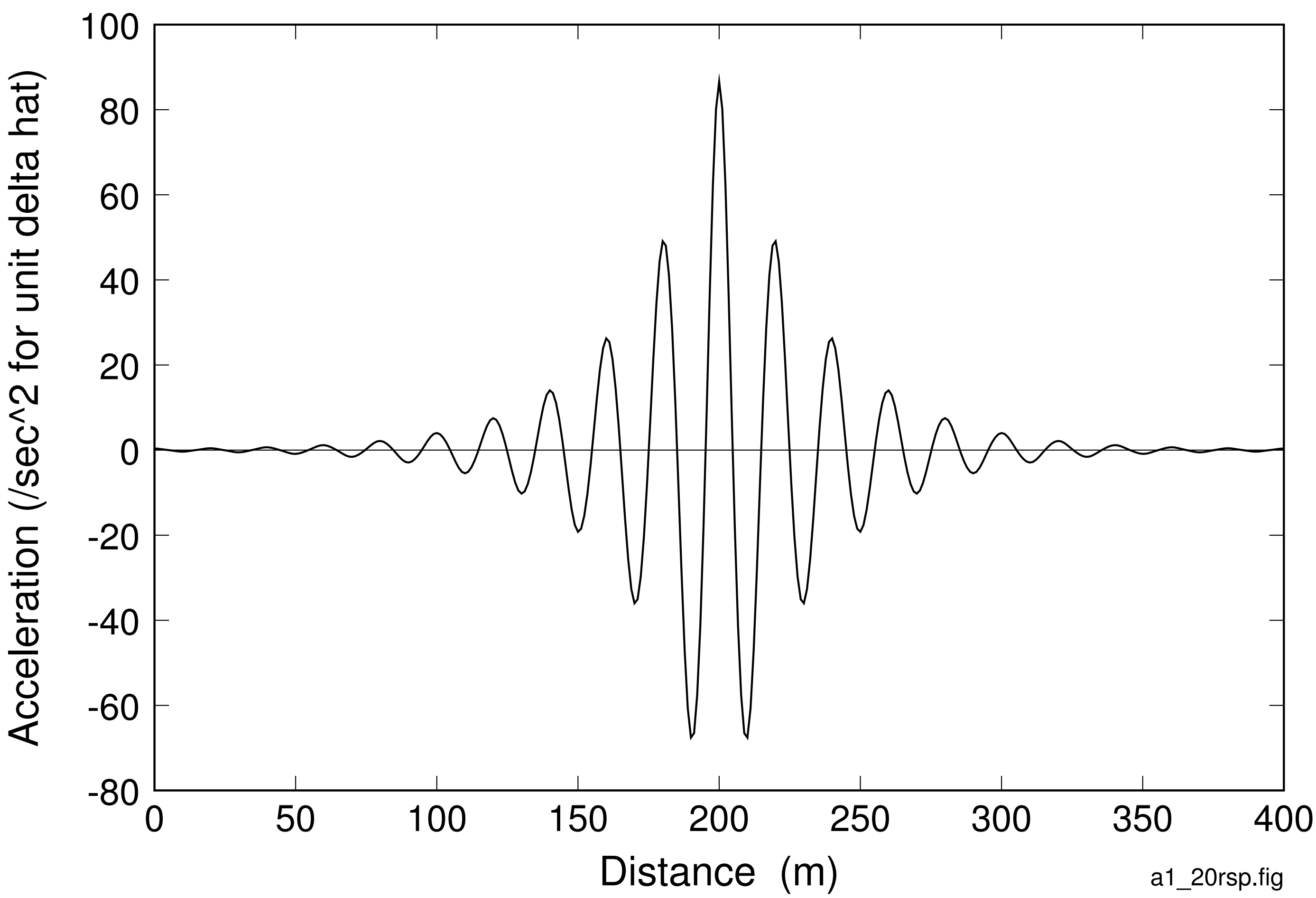


Figure 4. Vehicle acceleration response to road profile in Figure 3.

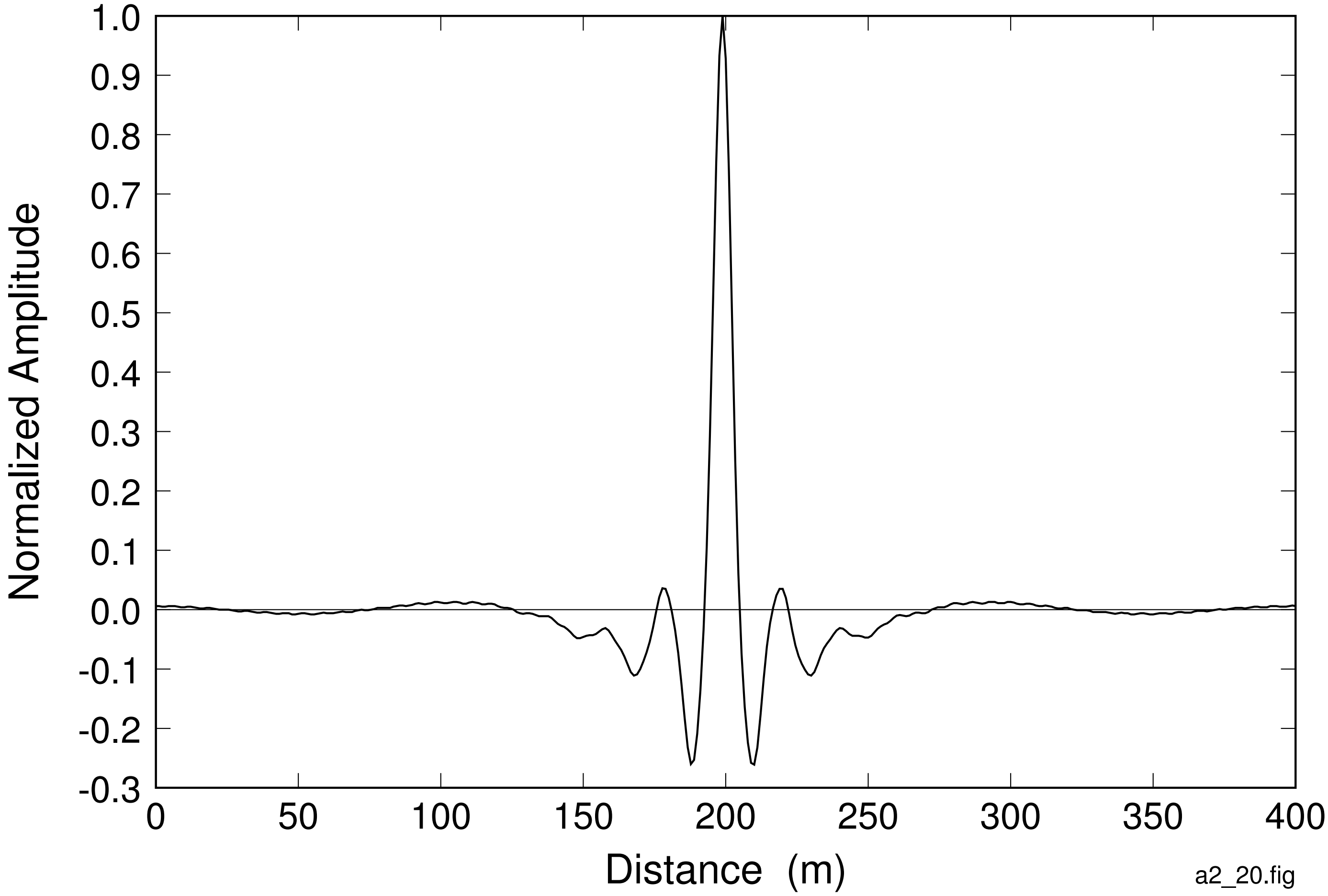


Figure 5. Worst-road for acceleration with $V = 20$ m/s and $\zeta = 0.2$.

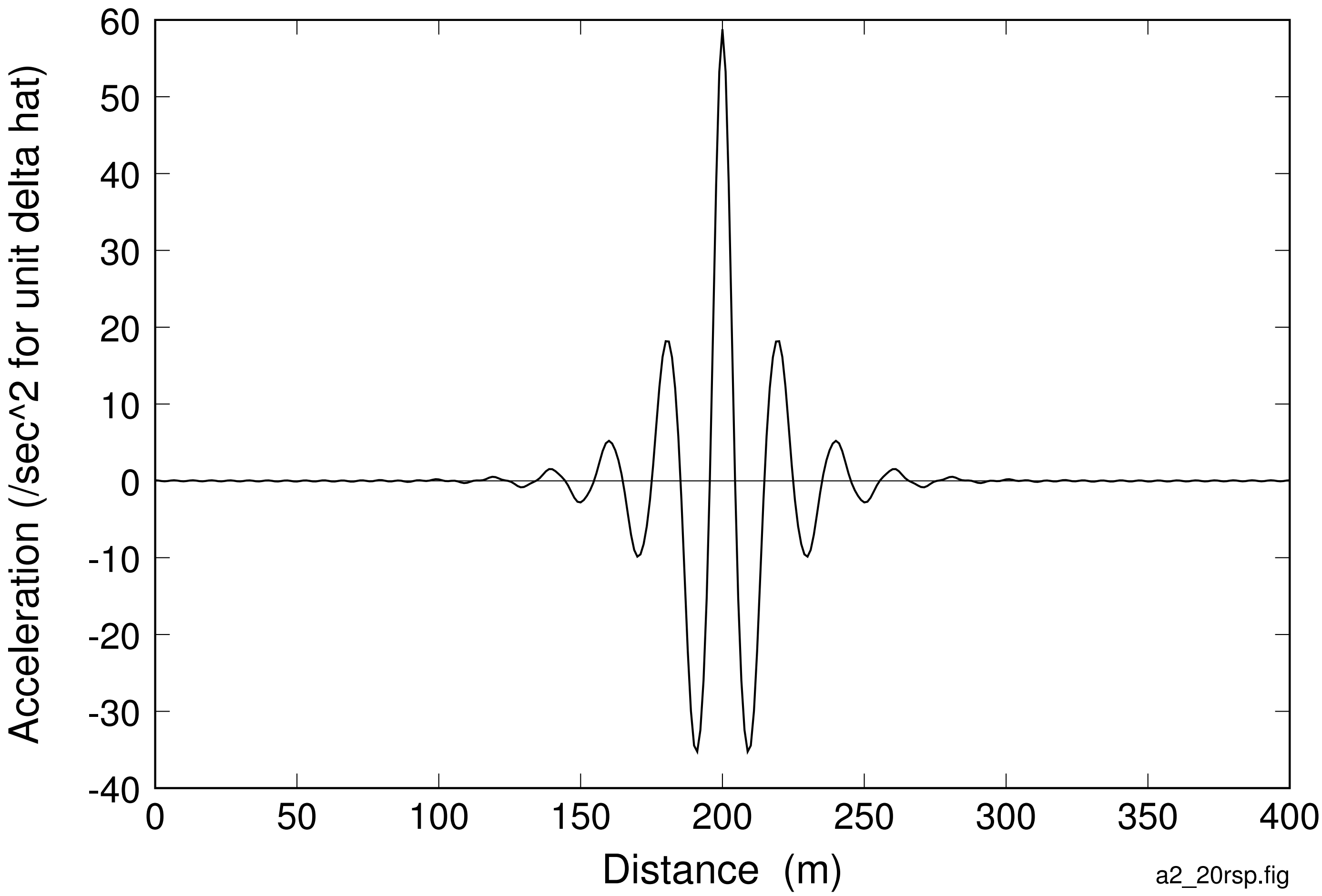


Figure 6. Vehicle acceleration response to road profile in Figure 5.

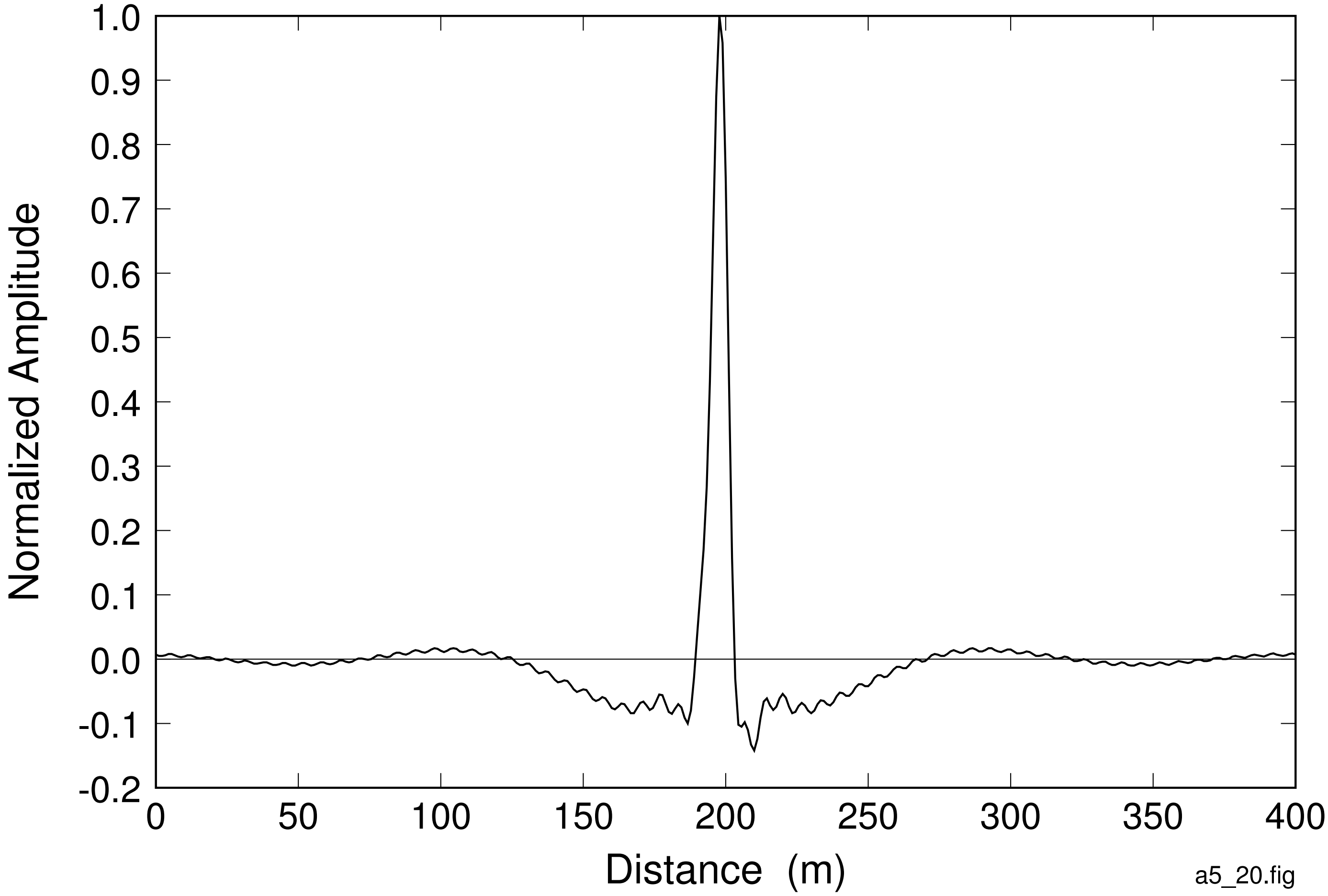


Figure 7. Worst-road for acceleration with $V = 20$ m/s and $\zeta = 0.5$.

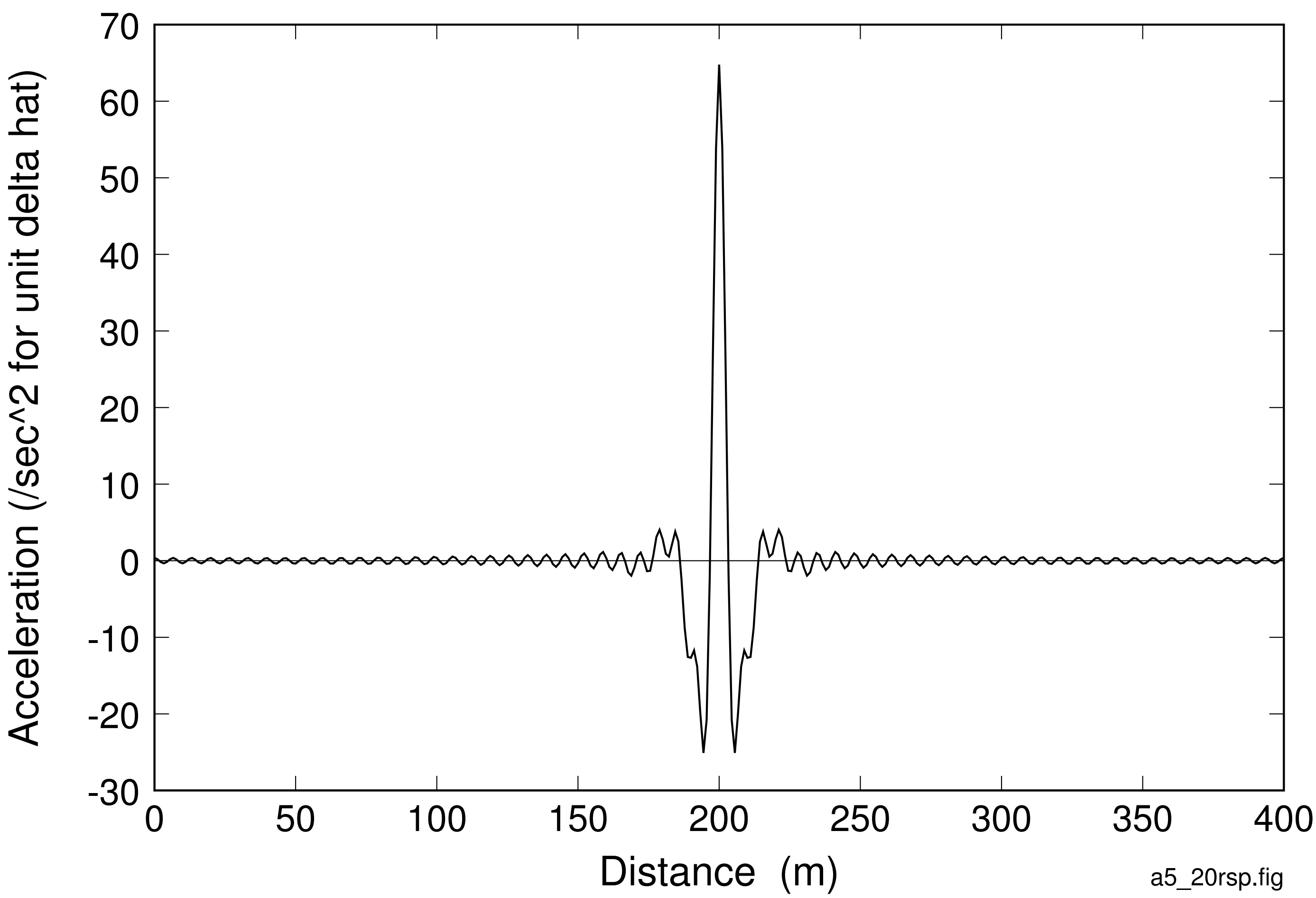


Figure 8. Vehicle acceleration response to road profile in Figure 7.

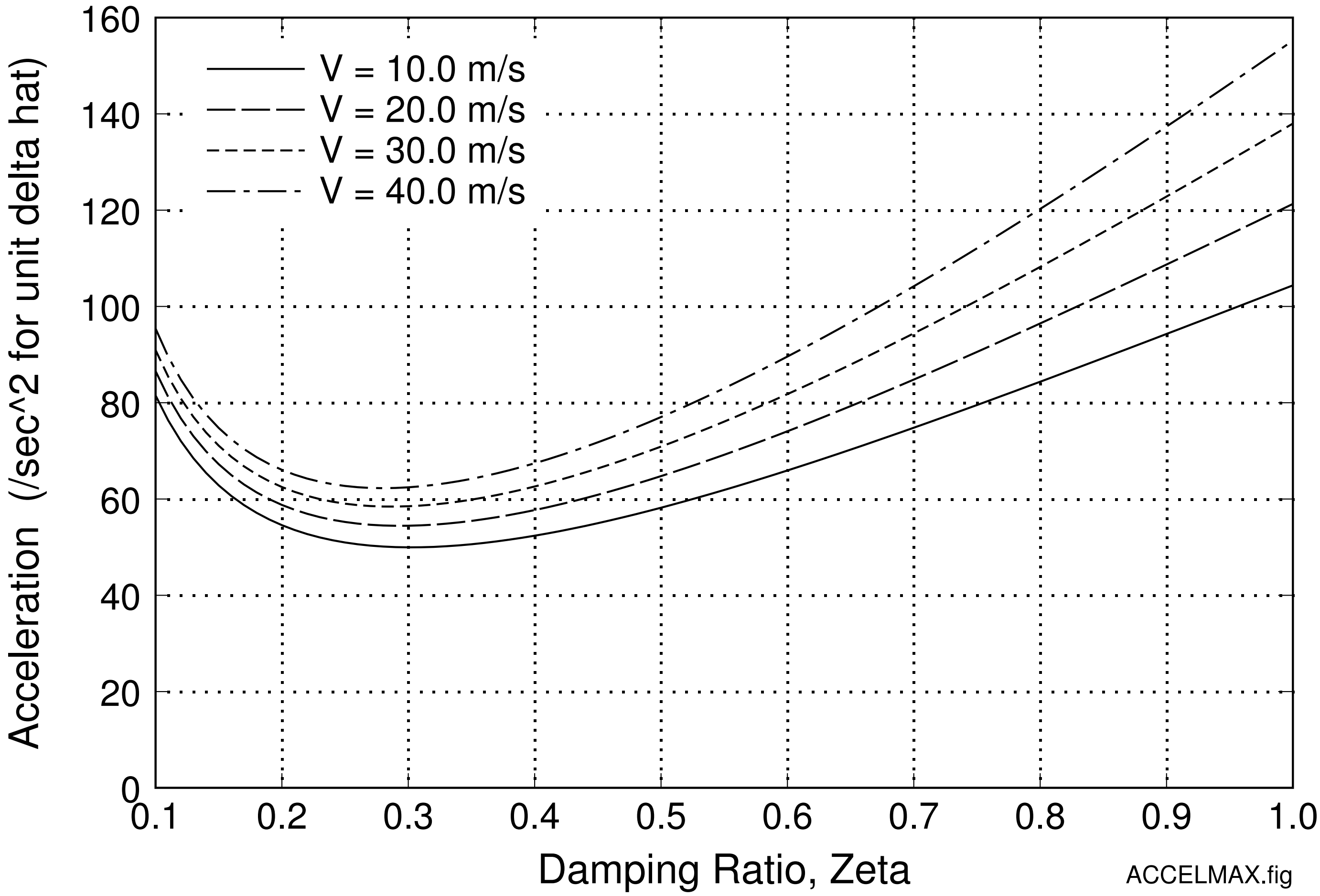


Figure 9. Vehicle acceleration maxima versus ζ for typical vehicle velocities (from 36 to 144 km/hr, about 22 to 90 mph).

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Conclusions and Recommendations

- Convex modelling is a useful adjunct to more conventional spectral response density analyses.
- The Fourier transform convex model gives road profiles that lead to undesirable response as well as maximum responses. These can be used as profiles to be avoided in road construction and to be included in test tracks to ensure extreme testing.
- The shape of the ellipsoid bound takes into account the process-dependent characteristics of machines and methods used in road construction, and road wear from vehicle transit and weathering. Thus, while the convex model focuses on worst-road profiles, the calculated profiles are based on measured wavelength dependence.
- Parameter studies lead to vehicle designs that minimize response to the worst roads for each particular vehicle.
- Convex modelling can be extended to the quarter-car model to determine worst-road profiles and response in the presence of wheel bounce as well as main body motion.